Important example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Facts: ① f is C^∞.

② All derivs of f are 0 at $x=0$
$\Rightarrow$ T.S. of f at $x=0$ is T.S. $= 0$.

③ $\therefore$ $f(x) = TS(x)$ only when $x=0$.

Note: f is certainly not analytic!

Nothing like this happens for holomorphic fcns, by the Taylor series Theorem.

We can use the Laurent series to examine singularities of a holomorphic function — if they are isolated.

Suppose $f: D \setminus \{z_0\} \to \mathbb{C}$ is a holomorphic fcn on a domain D, but f is not defined at $z_0 \in D$.

$B^*(z_0,R) = B(z_0,R) \setminus \{z_0\}$
$\subseteq D \setminus \{z_0\}$.

This is an annulus where

f is holomorphic $\Rightarrow$ $\exists!$ Laurent series $f(z) = \sum_{k\in\mathbb{Z}} a_k (z-z_0)^k$ that converges to f on $B^*(z_0,R)$.

Three possibilities

① $a_k = 0 \ \forall k < 0$.

In this case we call the singularity "removable".

② $a_m \neq 0$ for some $m < 0$ (could be several).
but $a_k = 0$ for $k < m$.

In other words, there are only a finite # of negative $(z-z_0)$ powers in the Laurent series.
(I.e. only a finite # of nonzero a_k with $k<0$.)

In this case, the singularity is called a "pole (of order $-m$)".

③ $a_k \neq 0$ for an infinite # of negative integers k

In this case, the singularity is called an essential singularity . .

Examples

① $f(z) = \dfrac{\sin(z)}{z}$ for $z \in \mathbb{C} \setminus \{0\}$.

Note $\sin(z) = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

$$\Rightarrow \frac{\sin(z)}{z} = \underbrace{\overset{k=0}{1} - \overset{k=2}{\frac{z^2}{3!}} + \overset{k=4}{\frac{z^4}{5!}} - \frac{z^6}{7!} + \cdots}_{\text{must be the Laurent series of } f(z)!} = \sum a_k z^k$$

Notice: No nonzero coefficients a_k with $k<0$!

$\therefore$ $z=0$ is a removable singularity of $\dfrac{\sin z}{z}$.

② $g(z) = \dfrac{\sin(z)}{z^{10}}$ ← holom on $\mathbb{C} \setminus \{0\}$.

We have $\sin(z) = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dfrac{z^9}{9!} - \cdots$ $\forall z \in \mathbb{C}$

$\Rightarrow$

$$\forall z \in \mathbb{C} \setminus \{0\}, \quad \frac{\sin(z)}{z^{10}} = z^{-9} - \frac{z^{-7}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-3}}{7!} + \frac{z^{-1}}{9!} - \frac{z}{11!} + \frac{z^3}{13!} - \cdots$$

$\therefore$ $z=0$ is a pole of order 9.

③ $h(z)=$ $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \ \forall z \in \mathbb{C}.$

$$\sin\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right) - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \cdots$$

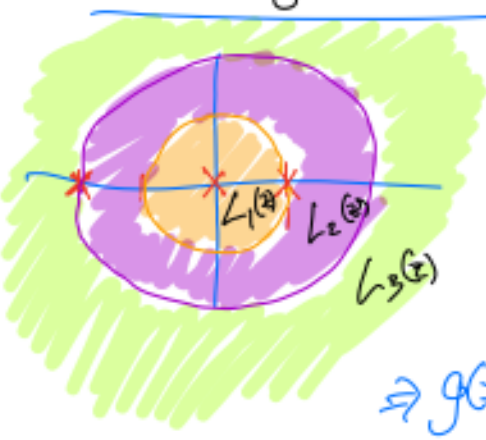
$\forall z \in \mathbb{C} \setminus \{0\}$

$$\sin\left(\frac{1}{z}\right) = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots$$

This $z=0$ is an essential singularity of $\sin\left(\frac{1}{z}\right)$.

Very important: must choose the Laurent series for the closest annulus (ie $B^*(z_0,R)$) to determine the type of singularity.

$\boxed{\text{Example}}$ Consider $g(z) = \frac{1}{z} - \frac{1}{z-1} + \frac{1}{z+2}$. Find all Laurent series centered at the origin.



First Laurent series $L_1(z)$. Good $0<|z|<1$

$$g(z) = \frac{1}{z} + \underbrace{\frac{1}{1-z}}_{|z|<1} + \underbrace{\frac{1}{2+z}}_{\frac{1}{2}\frac{1}{1-\left(\frac{-z}{2}\right)}}$$

$$\Rightarrow g(z) = \frac{1}{z} + \sum_{k\geq 0} z^k + \frac{1}{2}\sum_{k\geq 0}\left(\frac{-z}{2}\right)^k \quad \left|\frac{z}{2}\right|<1 \text{ ie } |z|<2$$

$$\Rightarrow \quad g(z) = \sum_{k\in\mathbb{Z}} a_k z^k,$$

when $a_{-1} = 1$, $a_k = 1 + \frac{(-1)^k}{2^{k+1}}$ for $k \geq 0$

$a_k = 0$ for $k < -1$.

$\therefore$ $g(z)$ has a pole of order 1 at $z = 0$.

$L_2(z)$: valid $1 < |z| < 2$

$$g(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2+z}$$

$$= \frac{1}{z} + \frac{1}{-z} \frac{1}{(1 - 1/z)} + \frac{1}{2(1-(-\frac{z}{2}))}$$

$\left|\frac{1}{z}\right| < 1 \Leftrightarrow |z| > 1$. ✓

$\left|\frac{-z}{2}\right| < 1 \Leftrightarrow |z| < 2$ ✓

$$L_2(z) = \frac{1}{z} + \frac{-1}{z} \sum_{m\geq 0} \left(\frac{1}{z}\right)^m + \frac{1}{2} \sum_{k\geq 0} \left(-\frac{z}{2}\right)^k$$

$$= \sum_{k\in\mathbb{Z}.} a_k z^k, \text{ where}$$

$a_{-1} = 0$

$a_{-m-1} = -1$ for $m \geq 0$.

$a_k = \frac{(-1)^k}{2^{k+1}}$ for $k \geq 0$.

$L_3(z) \to$ valid $|z| > 2 > 1$

$$L_3(z) = \frac{1}{z} - \frac{1}{z-1} + \frac{1}{2+z}$$

$$= \frac{1}{z} - \frac{1}{z(1-\frac{1}{z})} + \frac{1}{z}\left(\frac{1}{1-(\frac{-2}{z})}\right)$$

$\left|\frac{1}{z}\right| < 1 \Leftrightarrow |z| > 1.$ ✓ $\quad \left|\frac{-2}{z}\right| < 1 \Leftrightarrow |z| > 2.$ ✓

$$L_3(z) = \frac{1}{z} - \frac{1}{z}\sum_{r\geq 0}\underbrace{\left(\frac{1}{z}\right)^r}_{z^{-1-r}} + \frac{1}{z}\sum_{m\geq 0}\underbrace{\left(\frac{-2}{z}\right)^m}_{z^{-m-1}}$$

$= \sum_{k\in\mathbb{Z}} a_k z^k$, where

$-m-1 = k$

$-m = k+1$

$m = -k-1$

$$a_{-1} = 1$$

$$a_k = -1 + (-2)^{-k-1} \quad \text{for } k < -1$$

$$a_k = 0 \text{ for } k \geq 0.$$

Characteristics of the different kinds of singularities.

① Removable kind $f(z) = \sum_{k\geq 0} a_k (z-z_0)^k$ on $B^*(z_0, R)$.

In this case, we may let $g(z) = \begin{cases} f(z) & \text{for } 0 < |z-z_0| < R \\ a_0 & \text{for } z = z_0 \end{cases}$

Then $g(z) = \sum_{k\geq 0} a_k (z-z_0)^k$ on $B(z_0, R) \Rightarrow g$ is holomorphic with <u>no</u> singularity at z_0!

Example: $f(z) = \frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ on $\mathbb{C} \setminus \{0\}$.

Let $g(z) = \begin{cases} \frac{\sin(z)}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0. \end{cases}$

Then g is entire!

& It's Taylor series is $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

In all of the cases with removable singularities,

$$\lim_{z \to z_0} f(z) = a_0 = g(z_0).$$

i.e. this limit exists.

And the Laurent series is the Taylor series of $g(z)$ at $z = z_0$!

BTW $\Rightarrow$ This also implies that $f(z)$ is bounded in any (sufficiently small) $\overline{B(z_0, r)}$ centered at the removable singularity.